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Convexity and Hausdorff-Pompeiu Distance

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Abstract. The aim of this paper is to realize a decomposition of the usual convexity structures on metric spaces. Thus, a metric space is totally convex if and only if it satisfies the conditions (A) and (B) (Proposition 2). Also, it is totally externally convex if and only if both conditions (A′) and (B′) are satisfied (Proposition 4). Some connections between the convexity conditions (A) and (A′) and the Hausdorff-Pompeiu metric are investigated (see, for example, Corollary 3).

1. Notations and Preliminaries

Let (X,d) be a metric space, and let $S(x,r)=\{y\in X; d(x,y)<r\}$ and $S[x,r]=\{y\in X; d(x,y)\le r\}$ be the open and closed spheres in X of center x and radius r.

We begin with a simple remark. If (X,d) is the discrete metric space (i.e. $d(x,y)=1$ if $x\neq y$, and $d(x,x)=0$ for all $x\in X$), we have

$$S(x,r)=\overline{S(x,r)}=\begin{cases}\{x\}, & r\le 1\\ X, & r>1\end{cases} \qquad \text{and} \qquad S[x,r]=\begin{cases}\{x\}, & r<1\\ X, & r\ge 1.\end{cases}$$

One can see a discontinuity in the behaviour of these spheres at $r=1$. On the other hand, the spheres $S(x,r)$ and $S[x,r]$ vary continuously in both variables x and r, if (X,d) is the Euclidean space $\mathbb{R}^n$ or a normed space.

Let $CLB(X)$ be the space of nonempty closed and bounded subsets of X. The continuity of the functions $F: X\times\mathbb{R}_+^*\to CLB(X)$ (respectively $\overline{F}: X\times\mathbb{R}_+^*\to CLB(X)$) defined by $F(x,r)=S[x,r]$ (respectively $\overline{F}(x,r)=\overline{S(x,r)}$) is influenced by the above mentioned deficiency of the metric spaces. In general, it is necessary to assume that (X,d) satisfies certain conditions of convexity to guarantee the positive results concerning the continuity of these functions [3].

Our aim is to point out the role played by convexity type conditions on a metric space in this area. For this purpose we decompose the conditions of

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total convexity and total external convexity of a metric space into two other types of weaker convexity conditions.

Let us recall some concepts that will be needed later.

In [4, p.26], a metric space (X, d) is said to be *quasi-convex* (QC-spaces) if for each $\varepsilon > 0$ and each $a, b \in X$, $a \neq b$, there is $\delta > 0$ and a point $x \in X$ such that $d(a, x) < d(a, b) - \delta$ and $d(b, x) < \varepsilon$.

E. Blanc proved [4, p.27] the equivalence of the following conditions:

1. (X, d) is quasiconvex, and
2. $\overline{S(x, r)} = S[x, r]$, for all $x \in X$ and $r > 0$.

Obviously, in a QC-space we have $F = \overline{F}$.

The space (X, d) is said to satisfy the *convexity condition (A)*, if for each $x, y \in X$, $x \neq y$, and each $r, s > 0$ with $d(x, y) < r + s$ there exists $z \in X$ such that $d(x, z) < r$ and $d(y, z) < s$ [4, p.24].

The following properties are equivalent [4, p.46]:

1. (X, d) satisfies (A);
2. $S(S(E, \alpha), \beta) = S(S[E, \alpha], \beta) = S(E, \alpha + \beta)$, for $E \subset X$ and $\alpha, \beta > 0$;
3. $S[S(E, \alpha), \beta] = S[S[E, \alpha], \beta] = S[E, \alpha + \beta]$, for $E \subset X$ and $\alpha, \beta > 0$.

Finally, we mention three well-known conditions of convexity.

(M) (*convexity in the sense of Menger*) for $x, y \in X$, with $d(x, y) \neq 0$, there exists $z \in X, x \neq z \neq y$, such that $d(x, y) = d(x, z) + d(z, y)$;

(C) (*total convexity*) for $x, y \in X$, with $d(x, y) \neq 0$, and $\alpha \in (0, d(x, y))$, there exists $z \in X$ such that $d(x, z) = \alpha$ and $d(x, z) + d(z, y) = d(x, y)$;

(C′) (*total external convexity*) for $x, y \in X$, with $d(x, y) \neq 0$, and $\alpha > 0$, there exists $z \in X$ such that $d(y, z) = \alpha$ and $d(x, y) + d(y, z) = d(x, z)$.

It is easy to check that the implications (C) $\Rightarrow$ (A) $\Rightarrow$ (QC), and (C) $\Rightarrow$ (M) are true.

2. DECOMPOSITION OF THE TOTAL CONVEXITY CONDITION

We begin with a characterization of the condition (A).

Proposition 1. *Let (X, d) be a metric space. The following are equivalent:*

(i) *X satisfies (A),*

(ii) *for any $x, y \in X$ and $r > 0$ we have*

$$d(x, S[y, r]) = \begin{cases} 0, & x \in S[y, r] \\ d(x, y) - r, & x \notin S[y, r]. \end{cases} \tag{1}$$

Proof. (i) $\Rightarrow$ (ii). Assume that $x \notin S[y, r]$. For $u \in S[y, r]$ we have

$$d(x, u) \geq d(x, y) - d(y, u),$$

hence

$$(2) \qquad d(x,u) \geq d(x,y) - r, \qquad \text{for all} \quad u \in S[y,r].$$

On the other hand, for any $\varepsilon > 0$ we have $d(x,y) < d(x,y)+\varepsilon < (d(x,y)-r+\varepsilon)+r$, and, by condition (A), there exists $x_\varepsilon \in X$ such that $d(x,x_\varepsilon) < d(x,y)-r+\varepsilon$ and $d(y,x_\varepsilon) < r$, i.e.,

$$(3) \qquad d(x,x_\varepsilon) < d(x,y) - r + \varepsilon, \quad \text{and} \quad x_\varepsilon \in S[y,r].$$

By (2) and (3) we deduce that

$$\inf\{d(x,u); u \in S[y,r]\} = d(x,y) - r,$$

therefore $d(x, S[y,r]) = d(x,y) - r$, for $x \notin S[y,r]$.

(ii) $\Rightarrow$(i). First of all, we observe that, if "$<$" is replaced by "$\leq$" in one or both inequalities $d(x,z) < r$ and $d(y,z) < s$ of (A), then we have a condition equivalent to (A).

Let $x, y \in X$, $x \neq y$, and $r, s > 0$ such that $d(x,y) < r + s$. We may assume, without any loss of generality, that $r, s < d(x,y)$. Then, by (ii), $d(x, S[y,s]) = d(x,y) - s$. Consequently, $d(x, S[y,s]) < r$, and there exists $z \in S[y,s]$ such that $d(x,z) < r$. So, there exists $z \in X$ such that $d(x,z) < r$ and $d(y,z) \leq s$, i.e., (A) is verified, and the proof is complete. □

Remark. 1) It is easy to see that if in the left side of (1), we place the open sphere $S(x,r)$, then the result remains valid.

2) Convexity in the sense of Menger and condition (A) are independent.

Indeed, let $X = [0,1] \times [0,1] \setminus \{(x,x); x \in (0,1)\}$ and d the Euclidean metric on X. It is easy to see that (X,d) verifies (A), but (M) is not satisfied (for the points $(0,0), (1,1) \in X$ is does not exists any point in X for which the required equality is valid).

For the reverse assertion, let us consider $X = [0,1) \cup (2,3]$ equipped with Euclidean metric. One can verify that (X,d) satisfies the property (M), but not (A). We leave the details to the reader.

Now, we need a new condition: the metric space (X,d) is said to satisfy the *condition* (B) if for any $x, y \in X$ and $r > 0$ there exists $z \in S[y,r]$ such that $d(x,z) = d(x, S[y,r])$.

The following assertion gives us a decomposition of the total convexity.

Proposition 2. *The metric space* (X,d) *is totally convex if and only if both conditions* (A) *and* (B) *are satisfied.*

Proof. Necessity is easy. For the sufficiency part, let x, y be two different points in X and $\lambda \in (0, d(x,y))$. Obviously, $y \notin S[x,\lambda]$. By (A), it follows that $d(y, S[x,\lambda]) = d(x,y) - \lambda$. Also, in view of (B), there exists a point $z \in S[x,\lambda]$ such that $d(y,z) = d(y, S[x,\lambda])$. Hence, in our hypotheses, we have

$$d(x,y) \leq d(x,z) + d(z,y) \leq \lambda + (d(x,y) - \lambda) = d(x,y).$$

Consequently, we obtain $d(x,z) = \lambda$ and $d(x,y) = d(x,z) + d(z,y)$, and the total convexity of (X,d) is established. □

3. Decomposition of the Total External Convexity Condition

We use a procedure analogous to that in Section 2.

Let us start with two definitions.

The metric space (X,d) is said to satisfy the *condition* (A′) if for each $x, y \in X$, $x \neq y$, and each positive real numbers r, s with $r - s < d(x,y) \leq r$ there is $z \in X$ such that $d(x,z) > r$ and $d(z,y) < s$.

Remark. 1) It can be easily verified that, if "$d(z,y) < s$" is replaced by "$d(z,y) \leq s$", we have a condition equivalent to (A′).
2) The condition (A) and (A′) have a simple geometric interpretation. (A) guarantees the existence of a common point of the open spheres $S(x,r)$ and $S(y,s)$ (or, equivalently, $S[x,r]$ and $S[y,s]$), whereas (A′) shows the existence of a point in $S(y,s)$ (or $S[y,s]$) which is external to the closed sphere $S[x,r]$.

Also, (X,d) is said to satisfies the *conditions* (B′) if for any $x, y \in X$ and $r > 0$ there exists $z \in S[y,r]$ such that $d(x,z) = \sup\{d(x,u); u \in S[y,r]\}$.

The following two propositions are similar to the Propositions 1 and 2.

Proposition 3. *Let (X,B) be a metric space. The following are equivalent:*

(i) *the condition (A′) is satisfied;*
(ii) *for any $x, y \in X$ and $r > 0$ we have*

$$\sup\{d(x,u); u \in S[y,r]\} = d(x,y) + r. \tag{4}$$

Proof. (i) ⇒ (ii). Indeed,

$$d(x,u) \leq d(x,y) + r, \quad \text{for all} \quad u \in S[y,r]. \tag{5}$$

Let $x \neq y$ and $\varepsilon > 0$ be given with $\varepsilon < r$. We can write $d(x,y) > d(x,y) - \varepsilon = (d(x,y) + r - \varepsilon) - r$ and so, by hypothesis, there exists $z \in X$ such that $d(x,z) > d(x,y) + r - \varepsilon$ and $d(y,z) < r$, that is

$$d(x,z) > (d(x,y) + r) - \varepsilon, \quad \text{and} \quad z \in S[y,r]. \tag{6}$$

Finally, combining (5) and (6), we obtain the relation (4).

(ii) ⇒ (i). Let $x, y \in X$, $x \neq y$, be given and let $r, s > 0$ be such that $r - s < d(x,y) \leq r$. By our hypothesis,

$$\sup\{d(x,u); u \in S[y,s]\} = d(x,y) + s < r,$$

hence there exists $z \in S[y,s]$ with $d(x,z) > r$, i.e., $d(x,z) > r$ and $d(y,z) \leq s$. Therefore, the condition (A′) is valid. This completes the proof. □

Proposition 4. *A metric space (X,d) is totally externally convex if and only if it satisfies the conditions (A′) and (B′).*

Proof. For the "if" part, let $x, y \in X$, $x \neq y$, and $\lambda > 0$ be given. In view of (A′), we have $\sup\{d(x,u); u \in S[y,\lambda]\} = d(x,y) + \lambda$. On the other hand, by (B′), there exists $z \in S[y,\lambda]$ such that $d(x,z) = \sup\{d(x,u); u \in S[y,\lambda]\}$.

Consequently, there is $z \in S[y,\lambda]$ such that $d(x,z) = d(z,y) + \lambda$. From $d(x,y) + \lambda = d(x,z) \leq d(x,y) + d(y,z) \leq d(x,y) + \lambda$, we deduce that $d(y,z) = \lambda$, and it follows that $d(x,z) = d(x,y) + d(y,z)$. Therefore X is totally externally convex.

The "only if" part is trivial. □

4. Hausdorff-Pompeiu Distance of Closed Spheres

In what follows, $e(A,B)$ and $H(A,B)$ denote the *excess* of A over B and the *Hausdorff-Pompeiu distance* between A and B, respectively. Under the conditions (A) and (A′) we will give a simple and natural expression for the Hausdorff-Pompeiu distance of two closed spheres. Now, we begin with a preliminary result.

Proposition 5. *Let (X,B) be a metric space. The following are true:*

(i) *condition* (A) *implies that if $S[x_1,r_1] \subset S[x_2,r_2]$, then*

$$e(S[x_1,r_1], S[x_2,r_2]) = 0,$$

and, if $S[x_1,r_2] \not\subset S[x_2,r_2]$, then

$$e(S[x_1,r_1], S[x_2,r_2]) \leq d(x_1,x_2) + r_1 - r_2 \tag{7}$$

(the cases $r_1 = 0$ and/or $r_2 = 0$ are not excluded);

(ii) *if the relation* (7) *holds, then X satisfies* (A).

Proof. (i) Let S_1 and S_2 denote the closed spheres $S[x_1,r_1]$ and $S[x_2,r_2]$, respectively. The first part is trivial. To show the second part of (i), we use Proposition 1. Indeed, if $S_1 \not\subset S_2$ we have

$$\begin{aligned} e(S_1,S_2) &= \sup\{d(x,S_2); x \in S_1\} = \sup\{d(x,S_2); x \in S_1 \backslash S_2\} = \\ &= \sup\{d(x,x_2) - r_2; x \in S_1 \backslash S_2\} \leq \\ &\leq \sup\{d(x,x_1) + d(x_1,x_2) - r_2; x \in S_1 \backslash S_2\} \leq \\ &\leq d(x_1,x_2) + r_1 - r_2. \end{aligned}$$

(ii) Assume that $x \in S[y,r]$. According to our hypothesis, we have

$$d(x, S[y,r]) = e(\{x\}, S[y,r]) = e(S[x,0], S[y,r]) \leq d(x,y) - r,$$

hence $d(x,S[y,r]) \leq d(x,y) - r$. Combining this fact with the relation (2) in Section 2, we obtain that $d(x,S[y,r]) = d(x,y) - r$. In view of Proposition 1, the space X fulfills (A). □

Corollary 1. *If the metric space (X,d) verifies the condition* (A)*, then*

$$H(S[x_1,r_1], S[x_2,r_2]) \leq d(x_1,x_2) + |r_1 - r_2|, \tag{8}$$

for all $x_1, x_2 \in X$ and $r_1, r_2 \in \mathbb{R}_+$.

Proof. In view of (7), we can write

$$\begin{aligned} H(S[x_1,r_1],S[x_2,r_2]) &= \max\{e(S[x_1,r_1],S[x_2,r_2]),e(S[x_2,r_2],S[x_1,r_1])\} \leq \\ &\leq \max\{d(x_1,x_2)+r_1-r_2,d(x_1,x_2)+r_2-r_1\} = \\ &= d(x_1,x_2)+|r_1-r_2|. \end{aligned}$$ □

Remark. The inequalities (7) and (8) can be strict. For example, in the Euclidean space $X=[0,1]$ we have

$$H(S[0,\frac{1}{2}],S[1,\frac{1}{2}]) = e(S[0,\frac{1}{2}],S[1,\frac{1}{2}]) = \frac{1}{2},$$

but $d(x_1,x_2)+|r_1-r_2| = 1+0 = 1$.

Corollary 2. *Let* (X,d) *satisfy condition* (A), *and* $CLB(X)$ *be equipped with the Hausdorff-Pompeiu uniformity* $\mathcal{U}_H$. *Then, the functions* F *and* $\overline{F}$ *(see Section 1) coincide and they are* $\mathcal{U}_H$*-uniformly continuous.*

Proof. According to our hypothesis, $\overline{S(x,r)} = S[x,r]$, for all $x \in X$ and $r>0$, hence $F=\overline{F}$.

Now, let $\varepsilon > 0$ be given. For any $x,x' \in X$ and $r,r' \in \mathbb{R}_+^*$ such that $d(x,x') < \frac{\varepsilon}{2}$ and $|r-r'| < \frac{\varepsilon}{2}$, we have

$$H(S[x,r],S[x',r']) \leq d(x,x')+|r-r'| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,$$

therefore F is $\mathcal{U}_H$-uniformly continuous. □

If we wish that (7) and (8) are valid with equality signs, then an additional hypothesis is needed.

Proposition 6. *In a metric space* (X,d) *the following are equivalent:*

(i) (X,d) *verifies* (A) *and* (A$'$);

(ii) *for any couple closed spheres in* X, *we have*

$$e(S[x_1,r_1],S[x_2,r_2]) = \begin{cases} 0, & S[x_1,r_1] \subset S[x_2,r_2], \\ d(x_1,x_2)+r_1-r_2, & S[x_1,r_1] \not\subset S[x_2,r_2]. \end{cases}$$

Proof. (i) $\Rightarrow$ (ii). As in the proof of the Proposition 5, we denote by S_1 and S_2 the closed spheres $S[x_1,r_1]$ and $S[x_2,r_2]$. If $S_1 \not\subset S_2$, in view of (A) we obtain, in the same way, that

$$\begin{aligned} e(S_1,S_2) &= \sup\{d(x,x_2)-r_2; x \in S_1 \backslash S_2\} = \\ &= \sup\{d(x_2,x); x \in S_1\} - r_2. \end{aligned}$$

Next, in the presence of (A$'$), and applying Proposition 3, we obtain

$$e(S_1,S_2) = d(x_1,x_2)+r_1-r_2,$$

which is the required equality.

(ii) $\Rightarrow$ (i). Let $x \in X$ and a closed sphere $S[y,r]$ be given. First, we prove (A). We observe that $e(\{x\},S[y,r]) = d(x,S[y,r])$. On the other hand, if

$x \notin S[y,r]$ and, by our hypothesis, we have $e(\{x\}, S[y,r]) = d(x,y) - r$. Hence, $d(x, S[y,r]) = d(x,y) - r$ if $x \notin S[y,r]$, and we can use Proposition 1 to conclude the proof of (A).

Let us prove (A′). We have

$$e(S[y,r], \{x\}) = \sup\{d(x,z); z \in S[y,r]\},$$

and, by the hypothesis,

$$e(S[y,r], \{x\}) = d(x,y) + r.$$

Thus,

$$\sup\{d(x,z); z \in S[y,r]\} = d(x,y) + r,$$

and, by Proposition 3, this is equivalent to (A′). □

Corollary 3. *In a metric space* (X,d) *which verifies the conditions (A) and (A′), the equality*

$$H(S[x_1,r_1], S[x_2,r_2]) = d(x_1,x_2) + |r_1 - r_2| \tag{9}$$

holds true for all $x_1, x_2 \in X$ *and* $r_1, r_2 \in \mathbb{R}_+$.

Let $\mathcal{S}$ be the set of all closed spheres of the metric space (X,d). Furthemore, on the set $X \times \mathbb{R}_+$ we define the metric δ by

$$\delta((x_1,r_1),(x_2,r_2)) = d(x_1,x_2) + |r_1 - r_2|.$$

Corollary 4. *If the metric space* (X,d) *verifies the conditions (A) and (A′), then the spaces* $(\mathcal{S}, H)$ *and* $(X \times \mathbb{R}_+, \delta)$ *are isometric.*

Proof. The function $f : \mathcal{S} \to X \times \mathbb{R}_+$ given by $f(S[x,r]) = (x,r)$ is an isometry of these spaces. □

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